LEBANESE AMERICAN UNIVERSITY
Division of Computer Science and Mathematics

# Calculus III

## Final Exam

Spring 2009 (June 22, 2009)

**Name:** Solutions

**ID:**

**Circle the name of your instructor:** Dr. Hamdan  Mr Puzantian:

| Question Number | Grade |
|---|---|
| 1. 8% | |
| 2. 6% | |
| 3. 16% | |
| 4. 6% | |
| 5. 7% | |
| 6. 7% | |
| 7. 6% | |
| 8. 16% | |
| 9. 6% | |
| 10. 16% | |
| 11. 6% | |
| Total | |

1. (8%) Determine if the following improper integrals converge or diverge:

(a) $\int_{15}^{\infty} \frac{1}{\sqrt{e^x - x^3}} dx$ $\sim$ $\int_{15}^{\infty} \frac{1}{e^{x/2}} dx < \int_{15}^{\infty} \frac{1}{x^{10}} dx$

converges:

p-series $p > 1$

(b) $\int_{-\infty}^{\infty} 4xe^{-x^2} dx$ $= \int_{-\infty}^{\infty} \frac{4x}{e^{x^2}} dx$ : odd fn.:

consider $\int_0^{\infty} \frac{4x}{e^{x^2}} dx = \int_0^{\infty} \frac{du}{e^u}$ converges.

$\Rightarrow$ the whole int. converges to $0$

2. (6%) Determine whether the following sequences$\{a_n\}$ converge or diverge:

(a) $a_n = \frac{\sin n}{\ln(n+1)!}$ $= \frac{\text{bdd}}{\infty} \longrightarrow 0.$

Converges to 0.

(b) $a_n = n \cos(1/n)$ $= n \cos 1/n \rightarrow \infty \cdot \cos 0 = \infty \cdot 1 = \infty$

$\therefore$ diverges

3. (16%) Determine whether the following series converge or diverge: In case of a converging alternating series, determine the type of convergence

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{7+n}$

$a_n = \frac{n}{7+n} \longrightarrow 1 \Rightarrow$

$a_n \not\to 0$

$\Rightarrow$ Series diverge. by $n^{\text{th}}$ term test

(b) $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}$

$a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} \longrightarrow e^{1} \Rightarrow$

$a_n \not\to 0 \Rightarrow$ Series conv. by $n^{\text{th}}$ term test.

(c) $\sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{3}-\frac{6}{n}\right)^n$

Root Test:

$\rho = \lim (a_n)^{1/n} \longrightarrow \frac{1}{3} - \frac{6}{n} \longrightarrow \frac{1}{3} \Rightarrow \rho = 1/3$

$\rho < 1 \Rightarrow$ Series conv. by Root test.

(d) $\sum_{n=1}^{\infty}\frac{n!}{6^{n+2}}$

Ratio Test

$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)!}{6^{n+3}} \cdot \frac{6^{n+2}}{n!} \longrightarrow \frac{n}{6} \to \infty$

$\rho = \infty > 1 \Rightarrow$ series Conv. by Ratio test.

4. (6%) For what values of $x$ does the series $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{n^2}$ converge?

$$\rho = \lim \left| \frac{2 \cdot (x-3)}{n^{2/n}} \right| \rightarrow \rho = |2(x-3)|$$

$$-1 < 2(x-3) < 1 \Rightarrow \frac{5}{2} < x < \frac{7}{2}$$

Check end pt:

1) $x = 5/2 \Rightarrow$ Series gives: $\sum \frac{(-1)^n}{n^2}$ conv

2) $x = 7/2 \Rightarrow \sum \frac{1}{n^2}$ ✓ $\Rightarrow$

Conv. for $\boxed{5/2 \le x \le 7/2}$

5. (7%) Find the McLaurin series for $f(x) = \frac{\ln(1+x^2)}{x}$. You may use series you are already familiar with.

$$\ln(1+t) = \int \frac{dt}{1+t} = \int 1 - t + t^2 - t^3 + t^4 \ldots$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \ldots$$

$$\therefore \ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$$

$$\frac{\ln(1+x^2)}{x} = x - \frac{x^3}{2} + \frac{x^5}{3} - \frac{x^7}{4} + \ldots$$

6. (7%) Evaluate, using power series the limit below: $\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9}$

$$\sin t = t - t^3/3! + t^5/5! - t^7/7! + \ldots$$

$$\sin(x^3) = x^3 - x^9/3! + x^{15}/5! - x^{21}/7! + \ldots$$

$$\sin(x^3) - x^3 = -x^9/3! + x^{15}/5! - x^{21}/7! + \ldots$$

$$\frac{\sin x^3 - x^3}{x^9} = \underbrace{-\frac{1}{3!}}_{\rightarrow \boxed{-1/6}} \underbrace{+ \frac{x^6}{5!} - \ldots}_{\rightarrow 0}$$

9. (6%) Find the following limit, if it exists:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^6+y^2}$ = Try $y = Kx^3$

$\lim \frac{k x^6}{2x^6 + k^2 x^6} \longrightarrow \frac{k}{2+k^2}$ depends

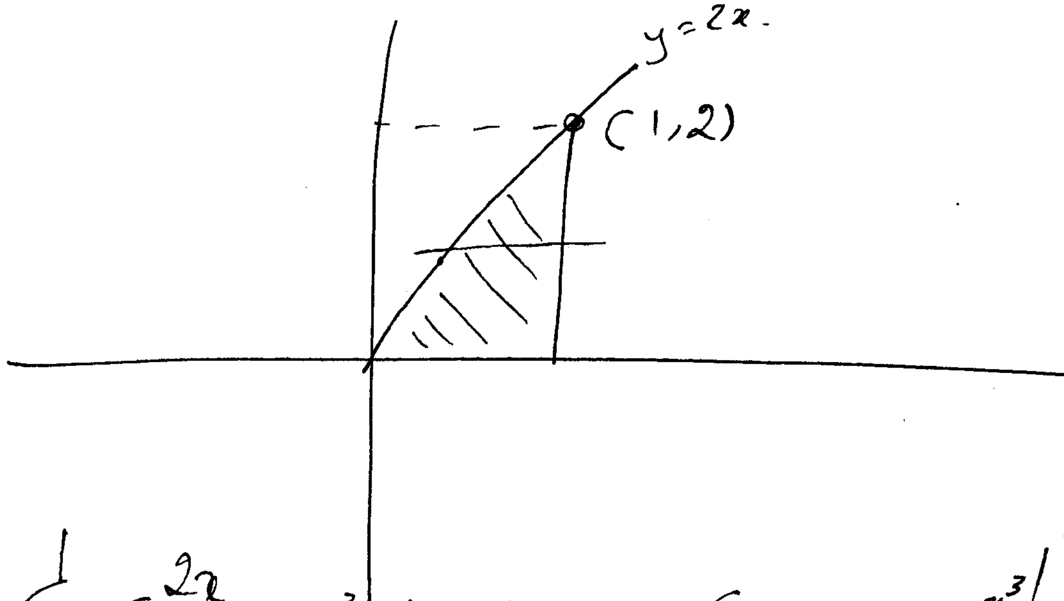
on $k$ $\rightarrow$ limit does not exist.

10. (16%) Evaluate the integrals:

(a) $\int_0^2 \int_{y/2}^1 ye^{x^3}\,dx\,dy$.(Hint: reverse the order of integration.)

$x = y/2$ $\quad 0 < y < 2$

$x = 1$



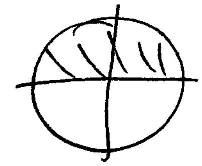

$= \int_0^1 \int_0^{2x} y e^{x^3}\,dy\,dx = \int \frac{y^2}{2} e^{x^3} \Big|_0^{2x} dx.$

$= \int 2x^2 e^{x^3}\,dx = \frac{2}{3} e^{x^3} \Big|_0^1$

$= \frac{2}{3}(e-1)$

(b) $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \frac{x^2y}{(x^2+y^2)^2}\,dydx$.(Hint: use polar coordinates)

$0 < y < \sqrt{1-x^2} \qquad -1 < x < 1$



$$\int_0^{\pi}\int_0^{1} \frac{r^3\cos^2\theta\,\sin\theta}{r^4}\; r\,dr\,d\theta.$$

$$\int_0^{\pi}\int_0^{1} \cos^2\theta\,\sin\theta\,dr\,d\theta$$ ✓ Can be done

11. (6%) Discuss in a short paragraph, using your own words, the relation between the following: $f_x(1,2)$ , $f_y(1,2)$ , the surface $(S) : z = f(x,y)$ and the planes $y = 2$, and $x = 1$.

Let S be the surface $z = f(x, y)$.

If we cut (S) by the plane $y = 2$,

then $f_x(1,2)$ = slope of the line

tangent to the curve $(S) \cap y=2$ @ $(2,1,f(2,1))$

Same for cutting by the plane $x = 1$ ...

$f_y(1,2)$ = slope of ...